

Renormalization of Extended QCD₂

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 Extended QCD (XQCD) proposed by Kaplan [1] is an interesting reformulation of QCD with additional bosonic auxiliary fields. While its partition function is kept exactly the same as that of original QCD, XQCD naturally contains properties of low energy hadronic models. We analyze the renormalization group flow of two-dimensional (X)QCD, which is solvable in the limit of large number of colors N_c , to understand what kind of roles the auxiliary degrees of freedom play and how the hadronic picture emerges in the low energy region.

Subject Index B00, B06, B32, B34, B35

1 Introduction

In Ref. [1], Kaplan proposed an interesting reformulation of QCD named as Extended QCD or XQCD. This new formulation contains additional auxiliary bosonic fields, keeping the partition function of QCD unchanged. The physics of XQCD is exactly the same as that of QCD, as long as the source operators of the ordinary quark and gluon fields are inserted.

It is shown in Ref. [1] that XQCD can describe several low energy hadronic pictures more naturally than QCD itself (in the limit of large number of colors N_c , where it is particularly simple to understand). The remarkable difference comes from the vacuum expectation value (VEV) of the auxiliary scalar field. This VEV directly gives the constituent mass to the quarks, which is an essential part of the quark models, and at the same time, makes the pseudo-scalar propagator massless, whose non-linear chiral transformation has exactly the same representation as the one in chiral perturbation theory. Moreover, it can be explained how the VEV is weakened by the presence of the baryonic source, the property suggested by the bag models [2].

The purpose of this paper is to understand what kind of roles auxiliary degrees of freedom play in the low energy region more concretely. It would be interesting if one could simulate lattice XQCD in four-dimensions and directly examine the above features. Unfortunately, the current formulation of XQCD suffers from the sign problem even with zero chemical potential. Instead, we study the two-dimensional version of (X)QCD (we will simply denote QCD₂ or XQCD₂ in the following), in the large N_c limit. This theory is known as the 't Hooft model [3] whose exact solution for quark propagator (in a particular gauge) and numerical solutions for meson masses given non-perturbatively. The advantage of studying the 't Hooft model is that the theory is particularly simplified in the large N_c limit and solvable. We consider this work as the first step to future studies of four-dimensional (X)QCD with $N_c = 3$.

In this work, we study the Wilsonian renormalization group (RG) flow of XQCD₂. We find that the auxiliary fields become dynamical when we take into account quantum corrections. Note that the degrees of freedom in XQCD should be the same as those in QCD, since auxiliary fields can give no effects on the original theory. Thus we can interpret the “dynamical auxiliary field” as just a transmutation of the degrees of freedom in QCD. In particular, the (pseudo)scalar auxiliary field should play a key role in the low energy effective action. It contains the degrees of freedom of pions, the lightest hadrons, as a consequence of the dynamical chiral symmetry breaking [4].

We also find that XQCD provides an interesting extension of the renormalization “scheme”. When we compute the RG flow, we usually restrict ourselves to the space of

the original fields given in our Lagrangian. In the case of QCD, for example, we only consider running of the couplings among quarks and gluons. However, in XQCD, we can insert at an arbitrary scale Λ_{cut} new bosonic degrees of freedom and the RG flow is extended to the space of their new interactions. Note that a similar idea was already tried in the works on the “dynamical hadronization” [5]. They converted the four-quark interactions, which were developed along the conventional RG flow of QCD, into the mesonic fields. But XQCD has a wider possibility in that no source of the original (four-quark) interaction is required. The scale(s) Λ_{cut} (’s) and the number of mesonic degrees of freedom are completely arbitrary. It is also important to note that XQCD has no risk of overcounting the physical degrees of freedom in original QCD.

This highly extended “scheme” of renormalization suggests many interesting applications beyond QCD. Since the number of auxiliary fields and its scale Λ_{cut} are arbitrary, “one” theory has infinitely many different effective actions at low energy, which are all physically equivalent. Moreover, the scheme suggests that there could exist a cut-off Λ_{cut} of the effective theory, which has no physical meaning. These aspects may give new insights to the current problems of the particle theory, such as naturalness problem. We would like to discuss these new possibilities in detail.

The rest of our paper is organized as follows. First, we review the formulation of XQCD including its two-dimensional version, and how it shows low energy hadronic pictures in the large N_c limit in Sec. 2. Then we explain our renormalization “scheme” in Sec. 3. Finally we compare the RG flow of QCD₂ and XQCD₂ in the large N_c limit. A summary is given in Sec. 6.

2 Extended QCD and its two-dimensional version

In this section, we review the original Extended QCD [1] in four dimensions and construct its two-dimensional version. We also summarize what is known in this two-dimensional large N_c QCD (the ’t Hooft model).

2.1 XQCD in four dimensions

We consider QCD with N_f flavors of quarks and gauge group $SU(N_c)$ in four-dimensional Euclidean spacetime :

$$S_{\text{QCD}} = N_c \int d^4x \left[\bar{\psi}_{ia} (\not{D} + m)^a_b \psi^{ib} + \frac{1}{4g^2} \text{Tr } \mathbf{F}_{\mu\nu} \mathbf{F}_{\mu\nu} \right], \quad (2.1)$$

where $\not{D} = \gamma^\mu (\partial_\mu + i\mathbf{A}_\mu)$ is the covariant derivative, and \mathbf{A}_μ denotes the gluon field. Here a, b, \dots are color indices and i, j, \dots are flavor indices.

XQCD is defined by introducing three types of auxiliary fields, the scalar field Φ , vector \mathbf{v}_μ and axial vector \mathbf{a}_μ , with the action in a Gaussian form,

$$S_{\text{aux}}[\Phi, \Phi^\dagger, \mathbf{v}_\mu, \mathbf{a}_\mu] = N_c \lambda^2 \int d^4x \left[\text{Tr} (\Phi^\dagger + 2\lambda^{-2} \bar{\psi}_a P_+ \psi^a) (\Phi + 2\lambda^{-2} \bar{\psi}_a P_- \psi^a) \right. \\ \left. + \frac{1}{2} \text{Tr} (\mathbf{v}_\mu + \lambda^{-2} \bar{\psi}_i \gamma_\mu \psi^i) (\mathbf{v}_\mu + \lambda^{-2} \bar{\psi}_i \gamma_\mu \psi^i) \right. \\ \left. + \frac{1}{2} \text{Tr} (\mathbf{a}_\mu + i\lambda^{-2} \bar{\psi}_i \gamma_\mu \gamma_5 \psi^i) (\mathbf{a}_\mu + i\lambda^{-2} \bar{\psi}_i \gamma_\mu \gamma_5 \psi^i) \right], \quad (2.2)$$

which keeps the original QCD partition function intact (up to a constant) :

$$Z_{\text{QCD}} = \int D\psi D\bar{\psi} D\mathbf{A}_\mu e^{-S_{\text{QCD}}[\psi, \bar{\psi}, \mathbf{A}_\mu]} \\ = \int D\psi D\bar{\psi} D\mathbf{A}_\mu D\Phi D\Phi^\dagger D\mathbf{v}_\mu D\mathbf{a}_\mu e^{-S_{\text{QCD}}[\psi, \bar{\psi}, \mathbf{A}_\mu] - S_{\text{aux}}[\Phi, \Phi^\dagger, \mathbf{v}_\mu, \mathbf{a}_\mu]} \\ \equiv Z_{\text{XQCD}}. \quad (2.3)$$

Here, the color singlet Φ transforms as a bifundamental representation under the $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry, and the flavor singlet \mathbf{v}_μ and \mathbf{a}_μ are $N_c \times N_c$ matrices (the singlet plus adjoint representations of the $SU(N_c)$ gauge group).

Note that each term of the action Eq. (2.2) has a non-renormalizable four-quark interaction. However, they automatically cancel through the Fierz identity

$$(P_+)_{mn}(P_-)_{m'n'} + (P_-)_{mn}(P_+)_{m'n'} = \frac{1}{4}[(\gamma_\mu)_{mn'}(\gamma_\mu)_{m'n} - (\gamma_\mu \gamma_5)_{mn'}(\gamma_\mu \gamma_5)_{m'n}], \quad (2.4)$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$. Therefore, our new theory ¹ :

$$S_{\text{XQCD}} = N_c \int d^4x \left[\bar{\psi}(\mathcal{D} + m)\psi + \frac{1}{4g^2} \text{Tr} \mathbf{F}_{\mu\nu} \mathbf{F}_{\mu\nu} \right. \\ \left. + \lambda^2 \left(\text{Tr} \Phi^\dagger \Phi + \frac{1}{2} \text{Tr} [\mathbf{v}_\mu \mathbf{v}_\mu + \mathbf{a}_\mu \mathbf{a}_\mu] \right) \right], \quad (2.5)$$

where

$$\mathcal{D} \equiv \not{D} + \not{\lambda} + i\not{a}\gamma_5 + 2(\Phi P_+ + \Phi^\dagger P_-), \quad (2.6)$$

is manifestly renormalizable.

Here, λ is an arbitrary parameter which has a mass dimension. Also, we can define the bare XQCD action at an arbitrary scale Λ_{cut} . Therefore, we have introduced two unphysical

¹ In general, scalar field Φ is a complex matrix. For $N_f = 2$, since the fundamental representation of $SU(2)$ is a pseudo-real representation, we can impose the reality condition $\Phi = \sigma_2 \Phi^* \sigma_2$ to Φ . In this case the factor $\frac{1}{2}$ is needed in front of the mass term of Φ .

scales. Of course, any physical observables cannot depend on λ nor Λ_{cut} . As explained below, the natural choice for the former value is the QCD scale, $\lambda \sim \Lambda_{\text{QCD}}$, while we want Λ_{cut} to be at higher energy near the real cut-off of the theory. Later we will discuss that this high ambiguity introduced in XQCD gives the extension of the “scheme” of the renormalization.

Since the integration over auxiliary fields is just a constant, the expectation value of any operator involving gluon and quark fields only, is equivalent to that of QCD :

$$\langle \mathcal{O}(\psi, \bar{\psi}, \mathbf{A}_\mu) \rangle_{\text{XQCD}} = \langle \mathcal{O}(\psi, \bar{\psi}, \mathbf{A}_\mu) \rangle_{\text{QCD}}. \quad (2.7)$$

This makes a big contrast to the previous attempts of simply adding scalar fields to QCD [6–8]. Since they are formally different from QCD, and have non-renormalizable four-quark interactions, it is non-trivial to keep the theory in the same universality class of QCD. In this respect, XQCD, which is exactly equivalent to QCD, has a theoretically firmer background.

Although XQCD and QCD are equivalent, their Feynman diagrams are quite different. The striking difference is seen when we assume a non-zero VEV to the chiral condensate. Since Φ shares the same quantum numbers as the scalar quark bilinear operator, it should also have VEV. In Ref. [1], it is explicitly computed in the large N_c limit as

$$\lambda^2 \langle \Phi_j^i \rangle_{\text{XQCD}} = -\delta_j^i \langle \bar{\psi}\psi \rangle_{\text{QCD}} \equiv \delta_j^i \Sigma. \quad (2.8)$$

This VEV directly gives the constituent mass $M = 2\Sigma/\lambda^2$ to the quarks.

It is also important to note the relative i between the \mathbf{v}_μ and \mathbf{A}_μ couplings in Eq. (2.6). It means that the exchange of \mathbf{v}_μ is repulsive while that of gluon is attractive. When the exchange of \mathbf{a}_μ is also taken into account, the repulsion is specifically between right-handed and left-handed quarks. Hence, the exchanges of vector and axial vector auxiliary fields (partially) weaken the attractive gluon exchanges. The introduction of the scalar auxiliary field Φ , which gives the constituent mass to the quarks, is concomitant with weakening of the interaction between quarks. This property is what assumed in the quark model [9, 10], described by weakly interacting massive quarks. In this way, XQCD naturally contains the feature of the quark model, which can not be explained by original QCD. As the ρ meson is made by two constituent quarks, an optimal choice [1] of λ is around 300 MeV.

Moreover, it is shown in [1] that the above quark model picture is compatible with the presence of the light pions as the (pseudo) Nambu-Goldstone (NG) bosons. Having the heavier constituent mass, the quark’s connected diagrams cannot have a long-range correlation. Instead, XQCD explicitly includes the propagation of Φ containing the pionic mode in it. Thus, XQCD diagrammatically distinguishes the pions from other mesons made by constituent quarks.

2.2 Application to the 't Hooft model

In this work, we consider the large N_c limit of QCD in two-dimensional Lorentzian space-time, which is the so-called 't Hooft model [3]. An exact solution for the quark propagator and numerical solutions for the meson masses are known. Solvability of the theory comes from the fact that gauge fields have only two degrees of freedom in two dimensions and we can eliminate the self-interaction of gauge fields by a suitable gauge fixing. The elimination of the self-interaction dramatically simplifies the theory in the large N_c limit. In addition to the simplicity, two-dimensional gauge theories show the confinement and the chiral symmetry breaking in the large N_c limit. Thus, the 't Hooft model is a good test ground for QCD. In this subsection, we briefly review this model and construct its extended version.

Let us first introduce the light-cone coordinate

$$x^\pm = (x^0 \pm x^1)/\sqrt{2}. \quad (2.9)$$

With this, the metric is given by

$$g^{+-} = g^{-+} = g_{+-} = g_{-+} = 1, \quad (2.10)$$

and all other components are zero. Note that $x^2 = x^\mu x_\mu = 2x^+x^-$.

Next, we take the light-cone gauge :

$$\mathbf{A}_- = \mathbf{A}^+ = 0. \quad (2.11)$$

This choice of gauge is Lorentz invariant, since its transformation is operated multiplicatively on each coordinate. With this gauge, the QCD Lagrangian is given in a simple form

$$\mathcal{L} = \frac{1}{2} \text{Tr} (\partial_- \mathbf{A}_+)^2 + \bar{\psi} i \not{\partial} \psi - m \bar{\psi} \psi - \frac{g}{\sqrt{N_c}} \bar{\psi} \mathbf{A}_+ \gamma^+ \psi. \quad (2.12)$$

Note that there is no self-interaction term among gluons.

Here γ^\pm are the gamma matrices satisfying

$$(\gamma^+)^2 = (\gamma^-)^2 = 0, \quad \{\gamma^+, \gamma^-\} = 2. \quad (2.13)$$

It is also useful to define

$$\gamma_3 = -\gamma^0 \gamma^1 = \frac{1}{2} [\gamma^+, \gamma^-], \quad (2.14)$$

which is the counterpart of γ_5 in four dimensions.

The Feynman rule is given by the gluon propagator, the vertex factor of the quark-antiquark-gluon interaction and the quark propagator,

$$D_{\mu\nu}(k) = i\delta_{\mu+}\delta_{\nu+}\frac{1}{(k_-)^2}, \quad (2.15)$$

$$-\frac{ig\gamma^+}{\sqrt{N_c}}, \quad (2.16)$$

$$\mathbf{S}_{\text{tree}}(p) = i \frac{p_+\gamma^+ + p_-\gamma^- + m}{2p_+p_- - m^2 + i\epsilon}. \quad (2.17)$$

Since every gluon-quark vertex contains γ^+ , the internal quark line is always sandwiched by two γ^+ s. Since

$$\gamma^+ \left\{ \begin{array}{c} 1 \\ \gamma^+ \\ \gamma^- \\ \gamma_3 \end{array} \right\} \gamma^+ = 2\gamma^+ \left\{ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right\}, \quad (2.18)$$

we only need to consider γ^- component of the quark propagator.

Thanks to this simple Feynman rule and the large N_c limit, we can non-perturbatively compute the quark self-energy. The quantum corrections to the quark propagator in the large N_c limit are expressed by the so-called rainbow diagrams shown in Fig. 1, which is proportional to γ^+ . Now the “full” quark propagator is expressed as

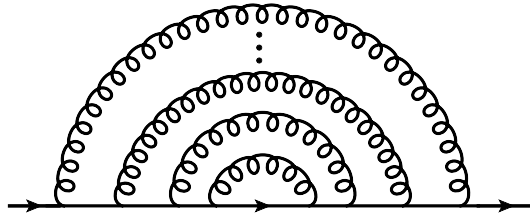


Fig. 1 A rainbow diagram contributing to quark propagator.

$$S(p) = \frac{ip_-}{2p_+p_- - m^2 - p_- \Sigma(p) + i\epsilon}, \quad (2.19)$$

and we obtain a self-consistent equation (see Fig. 2)

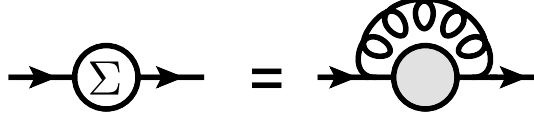


Fig. 2 A diagrammatic expression of the self-consistent equation for the self-energy $\Sigma(p)$.

$$-i\Sigma(p) = -4ig^2 \int \frac{dk_+ dk_-}{(2\pi)^2} S(p-k) \frac{1}{(k_-)^2}. \quad (2.20)$$

Note that the integration has the IR divergence at $k_- \rightarrow 0$. According to Ref [11], let us take the principle-value prescription and obtain

$$p_- \Sigma(p) = -\frac{g^2}{\pi}. \quad (2.21)$$

This result may look pathological since the constituent quark mass squared

$$M^2 = m^2 - g^2/\pi, \quad (2.22)$$

becomes tachyonic when g is strong. It is, however, regarded as just an artifact of the gauge fixing and the IR regularization. In fact, the other choice of the IR regularization, which give a positive values of M^2 , does not change the meson spectrum [12, 13].

To “extend” the ’t Hooft model is almost straightforward as the original XQCD in four-dimensions. However, there are two different points to be minded. One is the difference of the Fierz identity, which depends on dimensions. Another is the signature of the spacetime metric: to employ the light-cone gauge, we have to work in a Lorentzian spacetime, although original XQCD in Sec. 2 is defined in four-dimensional Euclidean spacetime.

The Fierz identity of two-dimensional theories is

$$(P_+)_{mn}(P_-)_{m'n'} + (P_-)_{mn}(P_+)_{m'n'} = \frac{1}{2}(\gamma_\mu)_{mn}(\gamma^\mu)_{m'n'}, \quad (2.23)$$

where γ^0 and γ^1 are taken to be hermitian and anti-hermitian respectively. Projection matrices P_\pm are defined by $P_\pm = (1 \pm \gamma_3)/2$. Note that there is no axial vector in two-dimensions. We can write the identity with quark fields such that

$$(\bar{\psi}_{ja} P_+ \psi^{ia}) (\bar{\psi}_{ib} P_- \psi^{jb}) = -\frac{1}{4} (\bar{\psi}_{ib} \gamma_\mu \psi^{ia}) (\bar{\psi}_{ja} \gamma^\mu \psi^{jb}). \quad (2.24)$$

Next, let us consider the auxiliary field path integral in the Lorentzian space-time,

$$\int \mathcal{D}\phi e^{iS(\phi)}. \quad (2.25)$$

Unlike the Euclidean case, it is not necessary for $S(\phi)$ to be positive because of the existence of the factor i . This means that there is some ambiguity in introducing the auxiliary fields.

In this work, we require a condition that the mass terms of the scalar and spacial part of the vector auxiliary fields are not tachyonic, at least, at the tree level, and then obtain

$$\begin{aligned}
e^{iS_{\text{aux}}[\Phi, \Phi^\dagger, \mathbf{v}_\mu]} &= \exp \left[-i\lambda^2 \int d^2x \left\{ \text{Tr} \left(\Phi^\dagger + \frac{\sqrt{2}}{\sqrt{N_c}} \frac{\alpha}{\lambda} \bar{\psi}_a P_+ \psi^a \right) \left(\Phi + \frac{\sqrt{2}}{\sqrt{N_c}} \frac{\alpha}{\lambda} \bar{\psi}_a P_- \psi^a \right) \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \text{Tr} \left(\mathbf{v}_\mu + \frac{1}{\sqrt{N_c}} \frac{\alpha}{\lambda} \bar{\psi}_i i\gamma_\mu \psi^i \right) \left(\mathbf{v}^\mu + \frac{1}{\sqrt{N_c}} \frac{\alpha}{\lambda} \bar{\psi}_i i\gamma^\mu \psi^i \right) \right\} \right] \\
&= \exp \left[i \int d^2x \left\{ -\frac{\alpha\lambda}{\sqrt{N_c}} \bar{\psi} [\sqrt{2}(\Phi P_+ + \Phi^\dagger P_-) - i\not{\mathbf{v}}] \psi - \lambda^2 \left(\text{Tr} \Phi^\dagger \Phi - \frac{1}{2} \text{Tr} \mathbf{v}_\mu \mathbf{v}^\mu \right) \right\} \right], \tag{2.26}
\end{aligned}$$

where λ and α are arbitrary real parameters. The mass dimensions of auxiliary fields and parameters are given by

$$[\Phi] = [\mathbf{v}_\mu] = 0, \quad [\alpha] = 0, \quad [\lambda] = 1. \tag{2.27}$$

The total action of XQCD₂ is given by

$$S_{\text{XQCD}} = \int d^2x \left[\bar{\psi} [\mathcal{D}' - m] \psi + \frac{1}{2} \text{Tr} (\partial_- \mathbf{A}_+)^2 - \lambda^2 \left(\text{Tr} \Phi^\dagger \Phi - \frac{1}{2} \text{Tr} \mathbf{v}_\mu \mathbf{v}^\mu \right) \right], \tag{2.28}$$

where

$$\mathcal{D}' \equiv i\not{\partial} - \frac{g}{\sqrt{N_c}} \mathbf{A}_+ \gamma^+ + \frac{i\alpha\lambda}{\sqrt{N_c}} \not{\mathbf{v}} - \frac{\sqrt{2}\alpha\lambda}{\sqrt{N_c}} (\Phi P_+ + \Phi^\dagger P_-). \tag{2.29}$$

In the above action, the mass term of quarks is the only source of explicit breaking of the chiral symmetry. We can absorb this symmetry breaking in the Φ 's shift:

$$\Phi \rightarrow \Phi - \frac{\sqrt{N_c}}{\sqrt{2}\alpha\lambda} m. \tag{2.30}$$

Then the fermion mass term is converted to

$$-m\bar{\psi}\psi \rightarrow \frac{\sqrt{N_c}}{\sqrt{2}} \frac{\lambda}{\alpha} m \text{Tr} (\Phi + \Phi^\dagger). \tag{2.31}$$

We also use this re-definition of the mass term in the RG studies of XQCD₂.

3 Extended renormalization scheme

As explained above, although QCD and XQCD are exactly equivalent, their low energy expressions are expected to be different. To understand this more clearly, we perform the Wilsonian renormalization group transformation on both theories and compare their low energy effective actions.

We would like to address two possible features of XQCD. One is how the mesonic degrees of freedom become dynamical. As Φ is expected to play a role of the NG boson at low energy, the RG flow should develop its kinetic term at low energy, keeping its mass near zero. Another issue is to see what happens on the original quark and gluon sectors along the RG flow. As hadrons play more important roles at low energy, the original quarks and gluons should decrease their relevance, and can eventually be decoupled from the effective action, near the scale of their (constituent) masses. We may be able to see this as cancellation with the vector auxiliary fields.

The inclusion of the auxiliary fields extends the (relevant) parameter space of the theory. The new terms of the effective Lagrangian we should consider are

$$\text{Tr} \partial_\mu \Phi^\dagger \partial^\mu \Phi, \quad \text{Tr} \partial_\nu \mathbf{v}_\mu \partial^\nu \mathbf{v}^\mu, \quad \text{Tr} (\partial_\mu \mathbf{v}^\mu)^2, \quad \text{Tr} \Phi^\dagger \Phi, \quad \text{Tr} \mathbf{v}_\mu \mathbf{v}^\mu, \quad \bar{\psi} (\Phi P_+ + \Phi^\dagger P_-) \psi, \dots \quad (3.1)$$

However, as the original theory has only two parameters g and m , the new interactions are not independent, but essentially controlled by these two parameters. Namely, the RG flows are restricted on a two-dimensional surface in the extended parameter space.

Which two-dimensional surface we take is determined by the choice of the regularization we use, and the re-definition of the coupling constants (by giving counterterms). In the view of RG flow of N parameters, $N - 2$ constraints can be given by these counterterms. Therefore, the choice of the surface corresponds to nothing but the choice of the renormalization scheme. Thus, XQCD can be regarded as the extension of the renormalization scheme to the extended theory space². The physics remains to be unchanged as the observables do not depend on the renormalization scheme.

In the conventional RG analysis, where we keep the original contents of the fields, the difference in the renormalization scheme means a tiny tuning of the paths of the (almost) fixed IR and UV points. For example, any scheme in QCD, sooner or later, eventually leads to the divergence of the gauge coupling and its effective Lagrangian becomes hard to analyze. However, the extended renormalization scheme, allowing the new field contents, provides us a wider choice of the effective actions. It is possible to have very different IR limits which share the exactly same physics. We already know some examples of such an equivalence as

² Note that extending the theory space and giving constraints on it, are widely used (sometimes unconsciously) even in the conventional RG analyses. For example, when we compute the renormalization of a supersymmetric theory, we have to employ some regularization which breaks the symmetry, and natural RG flows go through the non-supersymmetric space. We could still expect non-trivial cancellations of the contributions in that space, so that the theory remains to be supersymmetric. However, we usually do not take this strategy but instead make the theory back to the manifestly supersymmetric sub-space, by giving explicitly (or implicitly) counterterms which precisely cancel the appearance of non-supersymmetric terms. This can be done, at least perturbatively, unless the symmetry is anomalous.

“duality” [14]. It is an interesting question to ask if such a duality can be viewed as an example of the extended renormalization scheme.

In the following sections, we first perform the conventional RG transformation of two-dimensional QCD (QCD₂). Note that the model we take has the continuum limit, and the physical observables can be directly expressed by the bare parameters m and g . Since the non-perturbative solutions with m and g are already known, there is no need to perform RG transformations, other than comparing with XQCD.

Then, we introduce XQCD at a finite cut-off scale Λ_{cut} , and compare its RG flow with QCD, below that scale. As will be shown below, our computation uses a lot of approximations and assumptions. It is only at the one-loop level, employing a naive soft cut-off, assuming the convergence of the computation even in the Lorentzian space time, using truncations of the higher order Lagrangians, and so on. Nevertheless, we find that the RG flow of this simple model is theoretically non-trivial and interesting.

4 RG flow of QCD₂ in the large N_c limit

In this section, we analyze the RG flow of the 't Hooft model or QCD₂ itself, without introducing any auxiliary fields. As mentioned in the previous section, this theory is solvable with the bare Lagrangian in a well-defined continuum limit, and there is no practical needs to renormalize it. However, its RG analysis turns out to be quite instructive. Because of the small number of Feynman diagrams in the large N_c limit, we find that the counterterm which recovers the Parity symmetry, also recovers the gauge symmetry of the theory along the RG flow. Moreover, we find a non-perturbative “solution” (in a truncated theory space), which reasonably interpolates the theory in the continuum limit and that at the constituent quark mass. To our knowledge, such a non-perturbative analysis of RG flow in QCD₂ is not known before.

4.1 One-loop analysis and symmetry

Our goal is to integrate out the high energy modes of the quark and gluon fields in QCD₂ and obtain an effective action S_Λ at a finite cut-off Λ . If we could employ a gauge-invariant regularization, we expect that S_Λ has a similar form to the bare action :

$$S_\Lambda = \int d^2x \left[-\frac{1}{2} \text{Tr} (\mathbf{A}_+)_R \partial_-^2 (\mathbf{A}_+)_R + \bar{\psi}_R (i\not{\partial} - m_R(\Lambda)) \psi_R - \frac{g_R(\Lambda)}{\sqrt{N_c}} \bar{\psi}_R \mathbf{A}_+ \gamma^+ \psi_R + \cdots \right], \quad (4.1)$$

where $(\mathbf{A}_+)_R$ and ψ_R denote the renormalized fields, and $m_R(\Lambda)$ and $g_R(\Lambda)$ are the renormalized mass and coupling constant. If the effective action has this form, one can re-insert

the gauge degrees of freedom to the partition function and recover a manifestly gauge invariant form of the effective theory. Here we assume that our regularization smoothly cut off the high energy physics. In this work, we truncate the higher order terms and neglect irrelevant contributions at $O(1/\Lambda^4)$.

Since it is difficult to introduce the cut-off in a gauge covariant way, we usually lose the gauge invariance along the RG flow even in the truncated theory space. However, in QCD_2 in the light-cone gauge, thanks to the large N_c limit, the only the one term $:\bar{\psi}\partial_+\gamma^+\psi$ in the quark kinetic term obtains quantum corrections (see Fig. 3). Because of our choice of the light-cone gauge, this term breaks the Parity symmetry, and consequently breaks the gauge symmetry. By simply adding a counterterm or equivalently making a field transformation as we will see below, one can recover the Parity invariance of the theory, and the gauge symmetry as well.



Fig. 3 The one-loop correction to the quark propagator.

Let us demonstrate at the one-loop level how to obtain the effective action S_Λ from our bare action at $\Lambda = \infty$. It is obtained by expanding the weight $\exp(iS_{\Lambda=\infty})$ in the interaction terms, performing the higher momentum part above Λ of the loop integrals in advance, and re-exponentiating them to redefine the new action. In our case in the large N_c limit, we have only one term non-trivial in this high-mode integration and we obtain the one-loop result (in momentum space) as

$$\Delta S_\Lambda(\mathbf{A}_+, \psi, \bar{\psi}) = \int d^2p \left[-\bar{\psi}\gamma^+ \Delta\Sigma_\Lambda(p)\psi \right], \quad (4.2)$$

where

$$\Delta\Sigma_\Lambda(p) = 4g^2 \int \frac{d^2k}{(2\pi)^2} \left(1 - \frac{1}{R_A(-k^2/\Lambda^2)} \right) \frac{1}{(k_-)^2} \frac{(p_- - k_-)}{(p - k)^2 - m^2 + i\epsilon}. \quad (4.3)$$

Here, $R_A(-k^2/\Lambda^2)$ is a smooth function satisfying the boundary conditions

$$\lim_{k^2 \rightarrow \infty} \frac{1}{R_A(-k^2/\Lambda^2)} = 0, \quad \lim_{k^2 \rightarrow 0} \frac{1}{R_A(-k^2/\Lambda^2)} = 1. \quad (4.4)$$

Note here that we have not renormalized the theory, yet and the fields and coupling constants remain to be their bare values in the continuum limit.

From the Lorentz symmetry, this correction to the quark self-energy can be decomposed into two parts:

$$\Delta\Sigma_\Lambda(p) = -p_+A(p^2, \Lambda) + B(p^2, \Lambda)/p_-, \quad (4.5)$$

where A and B are regular functions in p^2 . The effective action is then

$$S_\Lambda = \int d^2p \left[\frac{1}{2} \text{Tr} \mathbf{A}_+ p_-^2 \mathbf{A}_+ - \frac{g}{\sqrt{N_c}} \bar{\psi} \mathbf{A}_+ \gamma^+ \psi + \bar{\psi} \{ p_- \gamma^- + p_+ \gamma^+ (1 + A(p^2, \Lambda)) - (m + \gamma^+ B(p^2, \Lambda)/p_-) \} \psi \right], \quad (4.6)$$

whose Parity symmetry is apparently lost. Moreover, one would have a concern about the IR behavior of the term $B(p^2, \Lambda)/p_-$.

However, we can remove these peculiar features by a simple field redefinition : defining $Z_\psi(p^2, \Lambda) = 1/(1 + A(p^2, \Lambda))$,

$$\psi \equiv \left(1 - \frac{\delta m(p^2, \Lambda)}{2p_-} \gamma^+ \right) Z_\psi(p^2, \Lambda)^{-\frac{\gamma^+ \gamma^-}{4}} \psi_t, \quad (4.7)$$

where $\delta m(p^2, \Lambda)$ is the greater solution of the equation

$$2B(p^2, \Lambda) = 2\delta m(p^2, \Lambda)m + \delta m(p^2, \Lambda)^2. \quad (4.8)$$

With this transformed field ψ_t , we obtain a desired form of the effective action,

$$S_\Lambda = \int d^2p \left[\frac{1}{2} \text{Tr} \mathbf{A}_+ p_-^2 \mathbf{A}_+ - \frac{g}{\sqrt{N_c}} \bar{\psi}_t \mathbf{A}_+ \gamma^+ \psi_t + \frac{1}{Z_\psi(p^2, \Lambda)} \bar{\psi}_t \left\{ p_- \gamma^- + p_+ \gamma^+ - \sqrt{Z_\psi(p^2, \Lambda)(m + \delta m(p^2, \Lambda))} \right\} \psi_t \right], \quad (4.9)$$

which has both of the Parity and gauge invariances. We can define the renormalized fields and couplings as

$$(\mathbf{A}_+)_R = \mathbf{A}_+, \quad \psi_R = \sqrt{1/Z_\psi(p^2, \Lambda)} \psi_t, \\ m_R(\Lambda) = \sqrt{Z_\psi(p^2, \Lambda)(m + \delta m(p^2, \Lambda))}, \quad g_R(\Lambda) = Z_\psi(p^2, \Lambda)g, \quad (4.10)$$

to obtain the effective action in Eq. (4.1). It is interesting to note that the apparent infrared singularity $B(p^2, \Lambda)/p_-$ is converted to the additive mass as the IR cut-off $\delta m(p^2, \Lambda)$. Also, note that the renormalization of the mass is not linear in $Z_\psi(p^2, \Lambda)$. These two facts indicate that the quantum correction cannot be considered as a simple quark's wave function renormalization.

To recover the gauge symmetry, the renormalization factor $Z_\psi(p^2, \Lambda)$ and the additive mass $\delta m(p^2, \Lambda)$ should not depend on p^2 . In the following computation, we will achieve this

by expanding these in p^2 around the constituent quark mass M , and set a renormalization condition around the point, stating that the higher order terms in $(p^2 - M^2)/\Lambda^2$ are irrelevant in the low energy region.

There is still one subtlety in the IR prescription of the gluon and quark fields. It is known that the light-cone gauge is sensitive not only to the IR regularization of the theory, but also to the UV regularization function R_A when it has a soft cut-off. Namely, the limit $\Lambda \rightarrow \infty$ and the functional integration may not commute, and the results may differ unless one carefully choose the IR structure of R_A . Having a mass gap in QCD, such an IR subtlety caused by massless gluons should be unphysical and have no effect on the physical observables. In fact, the previous works [3, 11] reported that the meson spectrum and other physical observables are insensitive to the choice of IR regularizations. In this work, however, we would like to keep the IR regularization of the gluon propagator unchanged from Ref. [11], in order to make the effect of the UV cut-off Λ clearer.

For the gluon propagator, following the prescription by Frishman [15] we define R_A by

$$\begin{aligned} \frac{i}{k_-^2 R_A(k^2/\Lambda^2)} = & 4i \frac{k_+^2}{k^2 + i\epsilon} \frac{1}{k^2 - \mu_{\text{IR}}^2 + i\epsilon} + \pi\epsilon(k_+) \left(\frac{-2k_+}{\mu_{\text{IR}}^2} \right) \left\{ \delta\left(k_- - \frac{\mu_{\text{IR}}^2}{2k_+}\right) - \delta(k_-) \right\} \\ & - 4i \frac{k_+^2}{k^2 + i\epsilon} \frac{1}{k^2 - \Lambda^2 + i\epsilon} - \pi\epsilon(k_+) \left(\frac{-2k_+}{\Lambda^2} \right) \left\{ \delta\left(k_- - \frac{\Lambda^2}{2k_+}\right) - \delta(k_-) \right\}. \end{aligned} \quad (4.11)$$

Here, the $\mu_{\text{IR}} \rightarrow 0$ limit has to be taken at the very end of the calculation. Note that we have introduced IR and UV cut-offs in a symmetric way, which makes our computation always IR finite, and the limit $\Lambda \rightarrow \infty$ and the path-integration commute. Without the second and fourth terms, the above prescription is similar to the conventional Pauli-Villars regularization of the gluon field. The second term corresponds to a homogeneous solution of equation of motion in the $\mu_{\text{IR}} \rightarrow 0$ limit.

Because of the above complication of the choice of IR regularizations, and Parity and gauge invariances, it is not a good idea to simply follow the standard procedure explicitly computing one by one, in particular, when one wants the computation beyond the one-loop. Since our bare action is well-defined and there are non-perturbative results in the continuum limit, it is much easier to start with the desired effective action Eq. (4.1) and compare the physical observables to those in the continuum limit, to determine the renormalized coupling and mass. Namely, in the following, we indirectly determine S_Λ by matching the functional integration from zero to infinity in the bare theory, and that from zero to Λ in the effective theory.

Now let us compute the (γ^- component of) quark propagator at the one-loop explicitly,

$$S_\Lambda(p) = \frac{ip_-}{p^2 - m_R(\Lambda)^2 - p_- \Sigma_\Lambda(p) + i\epsilon}, \quad (4.12)$$

where

$$\begin{aligned} \Sigma_\Lambda(p) &= 4g_R^2(\Lambda) \int \frac{d^2k}{(2\pi)^2} \frac{1}{k_-^2 R_A(k^2/\Lambda^2)} \frac{i(p_- - k_-)}{(p - k)^2 - m_R^2(\Lambda) + i\epsilon} \\ &= -\frac{1}{p_-} \left[\frac{g_R(\Lambda)^2}{\pi} + \frac{g_R(\Lambda)^2}{\pi} \left(\frac{p^2}{\Lambda^2} \log \left| \frac{\Lambda^2}{p^2} \right| + \frac{m_R(\Lambda)^2}{\Lambda^2} \log \left| \frac{\Lambda^2}{m_R(\Lambda)^2} \right| \right) \right] + O(1/\Lambda^4). \end{aligned} \quad (4.13)$$

Since the (non-perturbative) solution at $\Lambda = \infty$ is known [11],

$$S_\infty(p) = \frac{ip_-}{p^2 - M^2}, \quad M^2 = m^2 - \frac{g^2}{\pi}, \quad (4.14)$$

we can match this denominator with that of Eq. (4.12) up to a renormalization factor,

$$p^2 - m_R(\Lambda)^2 - p_- \Sigma_\Lambda(p) = Z_\psi(\Lambda)^2 (p^2 - M^2) \quad (4.15)$$

from which we can determine the renormalized quantities as

$$\begin{aligned} Z_\psi^2(\Lambda) &= \frac{1}{1 - \frac{g^2}{\pi\Lambda^2} \left(\log \left| \frac{\Lambda^2}{M^2} \right| - 1 \right)}, \\ g_R^2(\Lambda) &= Z_\psi^2(\Lambda) g^2 = \frac{g^2}{1 - \frac{g^2}{\pi\Lambda^2} \left(\log \left| \frac{\Lambda^2}{M^2} \right| - 1 \right)}, \\ m_R^2(\Lambda) &= m^2 \left(1 + \frac{2g_R^2(\Lambda)}{\pi\Lambda^2} \log \left| \frac{\Lambda^2}{M^2} \right| \right). \end{aligned} \quad (4.16)$$

4.2 Non-perturbative analysis

The above analysis can be easily extended to the non-perturbative level. The self-consistent equation for the rainbow diagram is given by

$$Z_\psi^2(p^2 - M^2) = p^2 - m_R^2 + \frac{g_R^2}{Z_\psi^2 \pi} + \frac{g_R^2}{Z_\psi^2 \pi \Lambda^2} \left(\frac{p^2}{\Lambda^2} \log \left| \frac{\Lambda^2}{p^2} \right| + \frac{M^2}{\Lambda^2} \log \left| \frac{\Lambda^2}{M^2} \right| \right). \quad (4.17)$$

Here we have omitted the arguments of the renormalized quantities for simplicity. We obtain a set of solutions as follows :

$$Z_\psi^2(\Lambda) = 1 + \frac{\frac{g^2}{\pi\Lambda^2} \left(\log \left| \frac{\Lambda^2}{M^2} \right| - 1 \right)}{1 - \frac{g^2}{\pi\Lambda^2} \log \left| \frac{\Lambda^2}{M^2} \right|}, \quad (4.18)$$

$$m_R^2(\Lambda) = m^2 \left(1 + \frac{\frac{2g^2}{\pi\Lambda^2} \log \left| \frac{\Lambda^2}{M^2} \right|}{1 - \frac{g^2}{\pi\Lambda^2} \log \left| \frac{\Lambda^2}{M^2} \right|} \right), \quad (4.19)$$

$$g_R^2(\Lambda) = \frac{Z_\psi^2(\Lambda)g^2}{1 - \frac{g^2}{\pi\Lambda^2} \log \left| \frac{\Lambda^2}{M^2} \right|}. \quad (4.20)$$

We find here that the chiral symmetry : $\lim_{m \rightarrow 0} m_R(\Lambda) = 0$ is not compatible with a simple relation for the coupling constant $g_R^2(\Lambda) = Z_\psi^2(\Lambda)g^2$. Since we want to keep the effective action chiral symmetric until very low energy limit, we have taken the former relation $m_R(\Lambda) \propto m$ as our renormalization condition.

The RG running of the mass and coupling constant are given in Fig. 4. Both of the renormalized parameters grow around the starting point as in four-dimensional QCD. However, it is interesting to note that they come back to the bare values around the scale of the constituent quark mass, which is consistent with the fact that their physical quantities around $\Lambda = M$ should be described by the bare values g and m again. For the scale below the constituent quark mass, there is a region where $g_R^2(\Lambda)$ and $m_R^2(\Lambda)$ go negative. We do not take this as a serious pathology but just a failure of our approximation in our crude analysis, including not taking the threshold effect carefully into account.

5 RG flow of XQCD₂ in the large N_c limit

Now let us investigate the RG flow of XQCD₂ in the large N_c limit. As in the previous section, we truncate our theory space to neglect $O(1/\Lambda^4)$ terms. Also, we require our effective action to be Parity and gauge invariant (let us just assume that our regularization keeps them by appropriate counterterms). The large N_c limit also helps to reduce some redundancy of the extended theory space. For example, the kinetic term of \mathbf{v}_μ is never developed. With

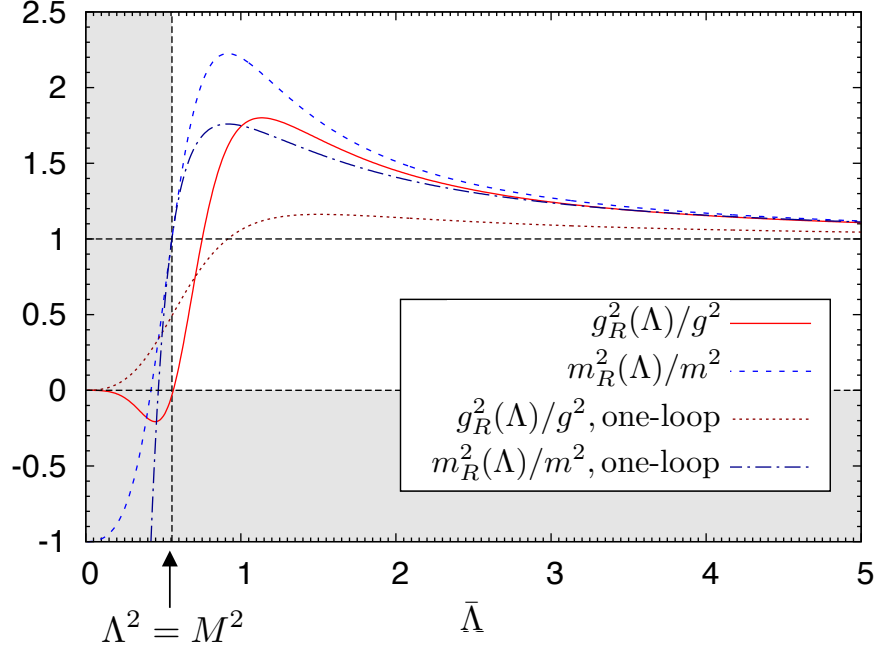


Fig. 4 The RG running of the mass and coupling of QCD₂. The solid curves are non-perturbative solutions, while the dashed ones are the one-loop results. The running coupling and mass do not monotonically increase but return to near the original bare values at $\Lambda \sim M$. Here, we make all quantities dimensionless using an arbitrarily chosen parameter Λ_0 , and use $\bar{\Lambda} = \Lambda/\Lambda_0$ for the horizontal axis. The bare parameters are set to $g/\Lambda_0 = 1$ and $m/\Lambda_0 = 0.1$.

this simplification, the most general form of the effective action is

$$\begin{aligned}
S_{\Lambda}^{\text{XQCD}} = \int d^2p \left[\frac{1}{2} \text{Tr} \mathbf{A}_+ p_-^2 \mathbf{A}_+ + \bar{\psi}_R [p - m_R(\Lambda)] \psi_R - \frac{g_R(\Lambda)}{\sqrt{N_c}} \bar{\psi}_R \mathbf{A}_+ \gamma^+ \psi_R \right. \\
+ Z_{\Phi}(\Lambda) \text{Tr} \Phi^\dagger p^2 \Phi - m_{\Phi}^2(\Lambda) \text{Tr} \Phi^\dagger \Phi - \frac{\sqrt{2}y(\Lambda)}{\sqrt{N_c}} \bar{\psi}_R (\Phi P_+ + \Phi^\dagger P_-) \psi_R \\
\left. + \frac{1}{2} \lambda^2 \text{Tr} \mathbf{v}_\mu \mathbf{v}^\mu + i \frac{\alpha \lambda}{\sqrt{N_c}} \frac{Z_{\psi}(\Lambda)}{Z_{\psi}(\Lambda_{\text{cut}})} \bar{\psi}_R \not{v} \psi_R \right]. \quad (5.1)
\end{aligned}$$

Neglecting the overall normalization of the fields, our theory space is extended from 2 (with m_R and g_R) to 5 dimensions (since α and λ do not run).

As discussed in Sec. 3, we can define a number of new RG schemes in this extended theory space, by choosing a two-dimensional surface in it. The simplest (and trivial) scheme is to take the three constraints :

$$Z_{\Phi}(\Lambda) = 0, \quad m_{\Phi}^2(\Lambda) = \lambda^2, \quad y(\Lambda) = \alpha \lambda, \quad (\text{at any } \Lambda), \quad (5.2)$$

along the RG flow. Note that three directions of five-dimensional space are fixed, and thus the RG flow is essentially two-dimensional. With this scheme, one can always integrate Φ and

\mathbf{v}_μ out and go back to original QCD₂ at any scale Λ . Since this scheme is exactly equivalent to the scheme in QCD₂, let us call it the “QCD scheme”.

We are interested in more non-trivial schemes, where the hadronic degrees of freedom become relevant (let us denote it the “hadronization scheme”). Let us require the same form of the constraints as Eq. (5.2) but only at a point $\Lambda = \Lambda_{\text{cut}}$:

$$Z_\Phi(\Lambda_{\text{cut}}) = 0, \quad m_\Phi^2(\Lambda_{\text{cut}}) = \lambda^2, \quad y(\Lambda_{\text{cut}}) = \alpha\lambda. \quad (5.3)$$

Then, the RG flows can go inside the bulk of the extended five-dimensional space. Notice that the space of our new RG flow still forms a two-dimensional surface, since it is forced to start from the two-dimensional surface at $\Lambda = \Lambda_{\text{cut}}$, and the RG equation is deterministic. In the following, we compute the RG flow of XQCD in this hadronization scheme and compare it with the QCD scheme.

5.1 One-loop analysis

Let us start with the computation at the one-loop. The three relevant diagrams in the large N_c limit are the quark self-energy (Fig. 3 the same as QCD₂), the Φ ’s self energy (Fig. 5), and Yukawa interaction (Fig. 6).

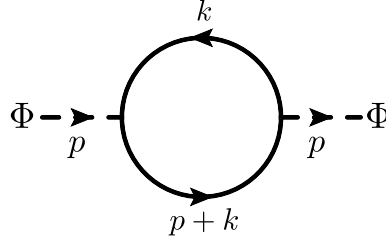


Fig. 5 Φ ’s self energy.

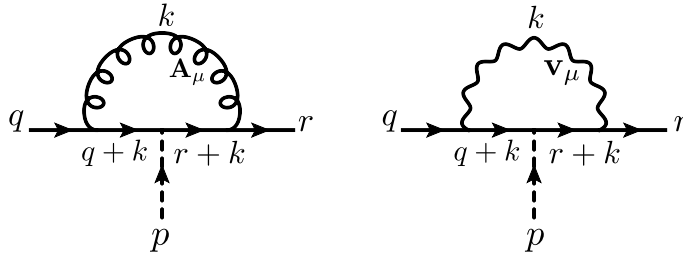


Fig. 6 Yukawa interaction.

Already at this moment, we can answer to our first question about the RG flow of the quark and gluon fields in XQCD₂. The three diagrams show that the scalar (and pseudo-scalar) Φ field receives quantum corrections from ψ and \mathbf{v}_μ , but never gives a feedback to them. Namely, the RG flow of the quark and gluon sector is unchanged. This result is not what we originally expected : weakening of the quark and gluon interactions. It seems that the two-dimension, the light-cone gauge, and the large N_c limit simplify the theory too much. We still expect a non-trivial difference in the case of four-dimensional QCD with $N_c = 3$.

Although there is no essential change in the RG flow of the quark mass and gauge coupling, the Feynman diagrams are quite different from those in original QCD. The essential change is in inclusion of the Yukawa interaction, which makes the mesonic degrees of freedom more relevant, as will be discussed below.

We begin with computing Φ 's self energy $\Pi(p)$ at the one-loop (Fig. 5). When we integrate out high momentum modes between two scales Λ and Λ_1 ($\Lambda > \Lambda_1$), we have

$$\begin{aligned} i\Pi(p) &= 2y^2(\Lambda) \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[P_+ \frac{i}{\not{k} - m_R(\Lambda)} P_- \frac{i}{(\not{p} + \not{k}) - m_R(\Lambda)} \right] \\ &\quad \times \left[\frac{1}{R_\psi(-k^2/\Lambda^2)} \frac{1}{R_\psi(-(p+k)^2/\Lambda^2)} - (\Lambda \leftrightarrow \Lambda_1) \right] \\ &= \frac{iy^2(\Lambda)}{\pi} \left[\left(\frac{1}{\Lambda_1^2} - \frac{1}{\Lambda^2} \right) \frac{5}{6} p^2 + \log \left(\frac{\Lambda}{\Lambda_1} \right) \right] + O(1/\Lambda^4, 1/\Lambda_1^4, m^2), \end{aligned} \quad (5.4)$$

where we have chosen the UV regulator

$$1/R_\psi(-k^2/\Lambda^2) = \frac{-\Lambda^2}{k^2 - \Lambda^2 + i\epsilon}. \quad (5.5)$$

These corrections are absorbed in the redefinition of $Z_\Phi(\Lambda_1)$ and $m_\Phi(\Lambda_1)$:

$$Z_\Phi(\Lambda_1) = Z_\Phi(\Lambda) + \frac{5y^2(\Lambda)}{6\pi} \left(\frac{1}{\Lambda_1^2} - \frac{1}{\Lambda^2} \right), \quad m_\Phi^2(\Lambda_1) = m_\Phi^2(\Lambda) - \frac{y^2(\Lambda)}{\pi} \log \left(\frac{\Lambda}{\Lambda_1} \right). \quad (5.6)$$

Next we turn to the computation of the Yukawa interaction (Fig. 6). The diagram on the left side of Fig. 6 is

$$\begin{aligned} &\frac{\sqrt{2}y(\Lambda)g_R^2(\Lambda)}{\sqrt{N_c}} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k_-)^2} \frac{2(q_- + k_-)m_R(\Lambda)}{(q+k)^2 - m_R^2(\Lambda)} \frac{1}{(r+k)^2 - m_R^2(\Lambda)} \\ &\quad \times \left[\frac{1}{R_A(-k^2/\Lambda^2)} \frac{1}{R_\psi(-(q+k)^2/\Lambda^2)} \frac{1}{R_\psi(-(r+k)^2/\Lambda^2)} - (\Lambda \leftrightarrow \Lambda_1) \right] \gamma^+. \end{aligned} \quad (5.7)$$

However, we neglect this contribution since it is of order $O(1/\Lambda^4)$.

The diagram on the right side of Fig. 6 is

$$\begin{aligned}
& \frac{\sqrt{2}y(\Lambda)\alpha_R^2(\Lambda)}{\sqrt{N_c}} \int \frac{d^2k}{(2\pi)^2} \gamma^\mu \frac{i}{\not{q} + \not{k} - m_R(\Lambda)} P_\pm \frac{i}{\not{r} + \not{k} - m_R(\Lambda)} \gamma^\mu \\
& \quad \times \left[\frac{1}{R_\psi(-(q+k)^2/\Lambda^2)} \frac{1}{R_\psi(-(r+k)^2/\Lambda^2)} - (\Lambda \leftrightarrow \Lambda_1) \right] \\
& = i \frac{\sqrt{2}y(\Lambda)}{\sqrt{N_c}} \frac{\alpha_R^2(\Lambda)}{\pi} \log\left(\frac{\Lambda}{\Lambda_1}\right) P_\pm + O(1/\Lambda^2, 1/\Lambda_1^2),
\end{aligned} \tag{5.8}$$

where P_\pm is P_+ or P_- when the dashed external line corresponds to Φ or Φ^\dagger respectively. $y(\Lambda_1)$ is defined by

$$y(\Lambda_1) = \frac{Z_\psi(\Lambda_1)}{Z_\psi(\Lambda)} \left[y(\Lambda) - \frac{y(\Lambda)\alpha_R^2(\Lambda)}{\pi} \log\left(\frac{\Lambda}{\Lambda_1}\right) \right], \tag{5.9}$$

Here, we have defined $\alpha_R(\Lambda) = \frac{Z_\psi(\Lambda)}{Z_\psi(\Lambda_{\text{cut}})} \alpha$.

We obtain the differential RG equations by setting $\Lambda_1 = \Lambda - d\Lambda$ in Eq. (5.6) and (5.9),

$$\begin{aligned}
\frac{dZ_\Phi(\Lambda)}{d\Lambda} &= \frac{5y^2(\Lambda)}{6\pi} \frac{d}{d\Lambda} \left(\frac{1}{\Lambda^2} \right), \\
\frac{dm_\Phi^2(\Lambda)}{d\Lambda} &= \frac{y^2(\Lambda)}{\pi} \frac{d}{d\Lambda} [\log(\Lambda)], \\
\frac{dy(\Lambda)}{d\Lambda} &= \frac{y(\Lambda)\alpha_R^2(\Lambda)}{\pi} \frac{d}{d\Lambda} [\log(\Lambda)] + \frac{y(\Lambda)g_R^2(\Lambda)}{2\pi} \frac{\partial}{\partial\Lambda} \left\{ \frac{1}{\Lambda^2} \left[\log\left(\frac{\Lambda^2}{M^2}\right) - 1 \right] \right\}.
\end{aligned} \tag{5.10}$$

At the lowest order of perturbation, the solutions of Eq. (5.10) are given by

$$Z_\Phi(\Lambda) = \frac{5y^2(\Lambda)}{6\pi} \left(\frac{1}{\Lambda^2} - \frac{1}{\Lambda_{\text{cut}}^2} \right) + O(\Lambda^{-4}), \tag{5.11}$$

$$m_\Phi^2(\Lambda) = \lambda^2 - \frac{y^2(\Lambda)}{\pi} \log\left(\frac{\Lambda_{\text{cut}}}{\Lambda}\right) + O(\Lambda^{-2}), \tag{5.12}$$

$$y(\Lambda) = \frac{\alpha\lambda}{1 + \frac{\alpha_R^2(\Lambda)}{\pi} \log\left(\frac{\Lambda_{\text{cut}}}{\Lambda}\right)} + O(\Lambda^{-2}), \tag{5.13}$$

where we have used the initial conditions Eq. (5.3).

As is expected, the Φ field becomes a dynamical variable, developing its kinetic term, as shown in Fig. 7.

5.2 Non-perturbative analysis

Since we have essentially only three types of planar diagrams, our computation of the RG flow can be, in principle, extended to a non-perturbative level. In particular, as sharing the

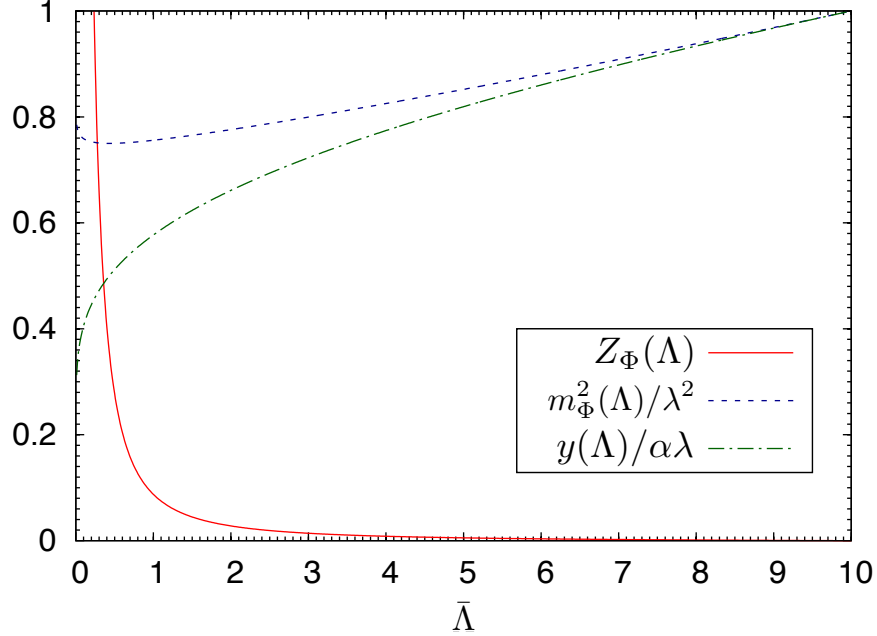


Fig. 7 RG running of the parameters of XQCD₂. In the same way as Fig. 4, all parameters are made dimensionless with the combination of the scale parameter Λ_0 . Here we set $\Lambda_{\text{cut}}/\Lambda_0 = 10$, $\alpha = 1$ and $\lambda/\Lambda_0 = 1$ as their initial conditions.

same quantum numbers as pions, we expect Φ to develop a massless pole in the pseudo-scalar channel.

Unfortunately, we find it not easy to confirm these expected features by simple loop computations even in the large N_c limit. In fact, this is a well-known problem of the light-cone gauge, which does not allow any gluonic correction to the scalar and pseudo-scalar vertices. Because of this simple structure, the chiral condensate is zero to all order of loop expansions in the light-cone gauge. However, the condensate in the 't Hooft model is known to be non-zero in the axial gauge [16], which is inconsistent with its gauge invariance. Although there have been several proposals [16–18] to give non-zero contribution from gluons to the scalar and pseudo-scalar vertices, the inconsistency is not yet solved completely as far as we know.

Here we do not go deep inside this controversial issue, but simply assume a non-zero expectation value of the chiral condensate in the $m \rightarrow 0$ limit [17] (we simply change our gauge to the axial gauge and come back to the light-cone gauge, assuming the full gauge invariance):

$$\langle \bar{\psi}\psi \rangle = -N_c \sqrt{\frac{g^2}{12\pi}}. \quad (5.14)$$

With this assumption, Φ has also a non-zero VEV (at $m \rightarrow 0$) given by

$$\langle \Phi \rangle = -\frac{1}{\sqrt{2N_c}} \frac{\alpha}{\lambda} \langle \bar{\psi} \psi \rangle = \sqrt{\frac{N_c}{24\pi}} \frac{\alpha g}{\lambda}. \quad (5.15)$$

Thus we may re-parametrize Φ as

$$\Phi = \langle \Phi \rangle e^{\frac{\sigma + i\pi}{\sqrt{2}}}, \quad (5.16)$$

where σ and π are $N_f \times N_f$ hermitian matrices. With this parametrization, we have

$$\text{Tr } \Phi^\dagger \Phi = \langle \Phi \rangle^2 \text{Tr } (1 + \sqrt{2}\sigma + \sigma^2 + \dots), \quad (5.17)$$

$$\text{Tr } (\Phi + \Phi^\dagger) = \langle \Phi \rangle \text{Tr} (1 + \sqrt{2}\sigma + \sigma^2/2 - \pi^2/2 + \dots). \quad (5.18)$$

The linear terms in σ in the above two contributions cancel out in the Lagrangian. Combining these equations with Eq. (2.31), the masses of σ and π are obtained by

$$m_\sigma^2 = \frac{\alpha^2}{N_c} \langle \bar{\psi} \psi \rangle^2 + O(m), \quad m_\pi^2 = \frac{1}{2} m \langle \bar{\psi} \psi \rangle + O(m^2). \quad (5.19)$$

Since the mass of π is proportional to the quark mass, it vanishes in the chiral limit $m \rightarrow 0$. This GMOR relation [19] is kept along the renormalization flow as long as our renormalization scheme preserves the chiral symmetry. For σ , its mass is proportional to Λ^2 since the mass $Z_\Phi^{-1}(\Lambda) m_\Phi^2(\Lambda)$ is proportional to Λ^2 . For the quarks, its mass is proportional to Λ since the Yukawa coupling $Z_\Phi^{-1/2}(\Lambda) y(\Lambda)$ is proportional to Λ . (see Subsec. 5.3 (2).) As we continue to integrate out high momentum modes, σ and quarks would decouple from the low energy dynamics at some scale, while π continues to contribute to the low energy dynamics. Eventually the theory is expected to go to the chiral effective theory described by the π field only and this confirms the low energy hadronic picture. We never reach this picture from the RG flow without auxiliary fields. In this way, the extension of the RG scheme introducing auxiliary fields gives a different aspect of the theory.

5.3 What is interesting in the extended RG flow ?

Here, we list interesting features and possible applications of the extended RG flow.

(1) Asymmetry in the RG flow of auxiliary fields

Along the RG flow, we have seen that \mathbf{v}_μ remains to be an auxiliary field since it receives no quantum correction in the large N_c limit. On the other hand, Φ acquires its kinetic term and becomes dynamical at the low energy. Clearly the RG flow of Φ and \mathbf{v}_μ is asymmetric. Since we can choose N_c and N_f differently, such an asymmetric RG flow is not special for the $N_c = \infty$ limit but should be common in

more general theories. This is not surprising since there is no symmetry between the two fields Φ and \mathbf{v}_μ .

Here, it is interesting to note that the cancellation of Φ and \mathbf{v}_μ auxiliary fields is manifest only at $\Lambda = \Lambda_{\text{cut}}$. If one only had the effective action at $\Lambda \ll \Lambda_{\text{cut}}$, it would be extremely difficult to identify that these Φ and \mathbf{v}_μ originally come from auxiliary fields. Equivalently, it would be difficult to see that this low energy limit of XQCD₂ is equivalent to QCD₂, unless one analyzes the high energy behavior around Λ_{cut} .

(2) *Fake UV divergence of auxiliary fields.*

In our analysis of the extended RG flow, we have not renormalized Φ so that its coefficient of the kinetic term to be different from unity. Here let us try the conventional canonical (re)normalization defining the renormalized field Φ_c by

$$\Phi_c \equiv \sqrt{Z_\Phi(\Lambda)}\Phi. \quad (5.20)$$

In terms of Φ_c , its effective mass and effective Yukawa coupling are $m_c(\Lambda) \equiv Z_\Phi^{-1}(\Lambda)m_\Phi^2(\Lambda)$ and $y_c(\Lambda) \equiv Z_\Phi^{-\frac{1}{2}}(\Lambda)y(\Lambda)$, respectively. In this normalization, as the renormalized scale Λ is approaching Λ_{cut} , both of the mass and Yukawa coupling diverge, since $Z_\Phi(\Lambda_{\text{cut}}) = 0$.

Even for Λ much smaller than Λ_{cut} , the effective mass and the Yukawa coupling behave as

$$\begin{aligned} Z_\Phi^{-1}(\Lambda)m_\Phi^2(\Lambda) &\sim \frac{6\pi\Lambda^2}{5y^2(\Lambda)} \left[\lambda^2 - \frac{y^2(\Lambda)}{\pi} \log \left(\frac{\Lambda_{\text{cut}}}{\Lambda} \right) \right], \\ Z_\Phi^{-\frac{1}{2}}(\Lambda)y(\Lambda) &\sim \sqrt{\frac{6\pi}{5}}\Lambda, \end{aligned} \quad (5.21)$$

which look still diverging: the mass diverges quadratically and the coupling diverges linearly when we go back the RG flow to high energy Λ .

We, of course, know that our theory is a super-renormalizable theory and has no divergence. The appearance of the fake divergence is simply due to the canonical normalization of the auxiliary degrees of freedom, and giving an infinite mass to Φ is consistent with the fact that the field Φ becomes a auxiliary field and decoupled from the theory.

However, suppose again one only knew the effective action at low energy $\Lambda \ll \Lambda_{\text{cut}}$. Then, one would find that this theory is very fine-tuned so that the UV divergence is precisely cancelled at $\Lambda = \Lambda_{\text{cut}}$ with another field \mathbf{v}_μ . Since \mathbf{v}_μ share no symmetry with Φ , and Λ_{cut} has no relation to the scale of the original theory, one could think of the cancellation as very “unnatural.”

(3) *Uniqueness of the “theory”*

One of the essential point of XQCD is that the introduction of auxiliary fields keeping the partition function unchanged. The key to achieve such a formulation is the Fierz identity (Eq. (2.4)), which allows two or three types of auxiliary fields cancelling each other. However, as discussed in the previous subsection, we have obtained quite different low energy effective actions by considering the RG flow of QCD and XQCD. In other words, we have two different descriptions for the same low energy theory.

Let us consider more radical set-ups. There exist infinitely large number of Fierz identities [20]. Moreover, the number of auxiliary fields and the scale Λ_{cut} are arbitrary. Namely, we have infinite number of the “extended” theories to describe one theory. Equivalently, we can say that the definition of one theory is not unique.

If there are infinitely many ways or path integrals to describe physics, why do we pick up one theory as the “standard” model ? Suppose a certain value of Λ_{cut} , and a certain number and kind of introduced auxiliary fields happened to make all the introduced auxiliary fields weakly coupled and precise computation of the observables quite easy. Then one would misidentify the formulation as a “unique” theory and discard other possible descriptions, unless one finds, by a lucky coincidence, a special re-formulation such as dualities. It is important to note that there is no physical meaning on Λ_{cut} , nor number and kind of auxiliary fields. The fact that we can introduce these unphysical scale(s), unphysical flavors, might give some hints for the long-standing problems in particle physics, like the hierarchy problem and problem of three generations.

(4) *UV completion for higher spin fields ?*

While \mathbf{v}_μ remains to be an auxiliary field in the large N_c limit, its kinetic term would appear in four-dimensional QCD with $N_c = 3$. In general, the UV completion of a massive vector field is not trivial. But in XQCD, the UV completion is quite obvious because the vector field reduces to the auxiliary field at Λ_{cut} . The extended RG flow naturally supply the UV completion of the massive vector fields. Since there are the Fierz identities whose corresponding fields contain higher spin fields [20], the extended RG flow might supply the UV completion of not only massive vector fields but also higher spin fields.

6 Summary

In this work, we have studied the RG flow of QCD_2 in the large N_c limit (the 't Hooft model) and its extension to XQCD_2 .

For QCD_2 , we have found the non-perturbative “solution” , which preserves the Parity symmetry and the gauge symmetry along the RG flow. As seen in Fig. 4, the values of the effective mass and coupling grow around the starting point and then return to the bare values around the scale of the constituent quark mass. We can see that the RG flow of QCD_2 smoothly interpolates the theory in the continuum limit and that at the constituent quark mass.

For XQCD_2 , although our specific analysis of the RG flow is at the one-loop level, we have found non-trivial and interesting pictures of the RG flow with auxiliary fields. By the introduction of the auxiliary fields, the parameter space of the theory is extended from the original one. However, as the auxiliary fields should not change the physics, the RG flow in the extended parameter space forms a surface whose dimension is the same as the original parameter space. The choice of the surface is not unique and corresponds to the choice of the (extended) renormalization scheme.

In Sec. 5, we have compared two schemes in the RG flow of XQCD . One is the “QCD scheme” where all auxiliary fields remain to be non-dynamical and equivalent to the RG flow of original QCD. They can be removed at any scale of Λ from the theory and we simply go back to the original QCD effective action.

Another is the “hadronization scheme”, where the scalar auxiliary field Φ becomes dynamical while the vector auxiliary field \mathbf{v}_μ still remains to be an auxiliary. Assuming the chiral symmetry breaking in the 't Hooft model, the constituent quarks and the massive scalar fields obtain a mass $\sim g$ or Λ . The only pions remain near massless and relevant in the low energy region. This confirms the hadronic picture of QCD. Since “QCD scheme” does not show this picture, we emphasize that we can never realize such a picture without taking into account the RG flow with auxiliary field, in other words, without adding the new elementary field which contains the pion degrees of freedom to QCD.

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